

Scattering problem for Klein-Gordon equation with cubic convolution nonlinearity *

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Abstract

The scattering problem for the Klein-Gordon equation with cubic convolution nonlinearity is considered. Based on the Strichartz estimates for the inhomogeneous Klein-Gordon equation, we prove the existence of the scattering operator, which improves the known results in some sense.

Keywords: Asymptotic of solution; Klein-Gordon equation; scattering operator

Subject class: 35P25, 81Q05, 35B05

1 Introduction

This paper is concerned with the scattering problem for the nonlinear Klein-Gordon equation of the form

$$\begin{cases} \partial_t^2 u - \Delta u + u = F_\gamma(u) & (t, x) \in R \times R^n \\ u|_{t=0} = f(x), \partial_t u|_{t=0} = g(x) \end{cases} \quad (1.1)$$

where u is a real-valued or a complex-valued unknown function of $(t, x) \in R \times R^n$. The nonlinearity is a cubic convolution term $F_\gamma(u) = -(V_\gamma(x) * |u|^2)u$ with $|V_\gamma(x)| \leq C|x|^{-\gamma}$. Here, $0 < \gamma < n$ and $*$ denotes the convolution in the space variables. The term $F_\gamma(u)$ is an approximative expression of the nonlocal interaction of specific elementary particles. The equation (1.1) was studied by Menzala and Strauss in [1].

In order to define the scattering operator for (1.1), we first give some Banach spaces. The usual Lebesgue space is given by $L^p = \{\phi \in S' : \|\phi\|_{L^p} < +\infty\}$, where the norm $\|\phi\|_{L^p} = \{\int_{R^n} |\phi(x)| dx\}^{1/p}$ if $1 \leq p < +\infty$ and $\|\phi\|_{L^\infty} = \sup_{x \in R^n} |\phi(x)|$ if $p = +\infty$. The weighted Sobolev space $H_p^{\beta,k}$ is defined by

$$H_p^{\beta,k} = \{\phi \in S' : \|\phi\|_{H_p^{\beta,k}} = \|\langle x \rangle^k \langle i\nabla \rangle^\beta \phi\|_{L^p} < +\infty\},$$

with $\langle x \rangle = \sqrt{1+x^2}$ and $\langle i\nabla \rangle = \sqrt{1-\Delta}$. We also write $H^{\beta,k} = H_2^{\beta,k}$ and $H^\beta = H_2^{\beta,0}$ if it does not cause a confusion. A Hilbert space $X^{\beta,k}$ is denoted by $H^{\beta,k} \oplus H^{\beta-1,k}$. Let $X_\rho^{\beta,k}$ be a ball of a radius $\rho > 0$ with a center in the origin in the space $X^{\beta,k}$. The scattering operator of (1.1) is defined as the mapping $S : X_\rho^{\beta,k} \ni (f_-, g_-) \rightarrow (f_+, g_+) \in X^{\beta,0}$ if the following condition holds:

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For $(f_-, g_-) \in X_\rho^{\beta,k}$, there uniquely exists a time-global solution $u \in C(R; H^\beta)$ of (1.1), and data $(f_+, g_+) \in X^{\beta,0}$ such that $u(t)$ approaches $u_\pm(t)$ in H^β as $t \rightarrow \pm\infty$, where $u_\pm(t)$ are solutions of linear Klein-Gordon equations whose initial data are (f_\pm, g_\pm) , respectively.

We say that $(S, X^{\beta,k})$ is well-defined if we can define the scattering operator $S : X_\rho^{\beta,k} \rightarrow X^{\beta,0}$ for some $\rho > 0$. In [2], Mochizuki prove that if $n \geq 3$, $\beta \geq 1$, $\gamma < n$ and $2 \leq \gamma \leq 2\beta + 2$, then $(S, X^{\beta,0})$ is well-defined. Hidano [3] see that if $n \geq 2$, $\beta \geq 1$, $4/3 < \gamma < 2$ and $k > 1/3$, then $(S, X^{\beta,k})$ is well-defined. By using the Strichartz estimate for pre-admissible pair and the complex interpolation method for the weighted Sobolev space, Hidano [4] shows that $(S, X^{\beta,k})$ is well-defined if $n \geq 2$, $\beta \geq 1$, $4/3 < \gamma < 2$ and $k > (2 - \gamma)/2$. Our aim of this article is to show that $(S, X^{\beta,1})$ is well-defined if $n \geq 2$,

$$1 < \gamma < \min\left\{\frac{2(n+1)}{n+2}, \frac{3n-2}{n+2}\right\}, \frac{(n+2)(\gamma+1)}{4n} + \frac{1}{2} < \beta < \frac{(n+2)(\gamma+1)}{2n}. \quad (1.2)$$

More precisely, we prove the following theorem.

Theorem 1.1 *Let the function $V_\gamma(x)$ satisfy*

$$|V_\gamma(x)| \leq C|x|^{-\gamma}, \quad |\nabla V_\gamma(x)| \leq C|x|^{-(1+\gamma)}.$$

Assume that $n \geq 2$, γ and β satisfy (1.2). Then there exists a positive number $\delta_0 > 0$ satisfying the following properties:

(1). *For $(f, g) \in X^{\beta,1}$ with $\|(f, g)\|_{X^{\beta,1}} \leq \delta_0$, there uniquely exist final states $(f_\pm, g_\pm) \in X^{\beta,0}$ and a solution $u(t) \in C(R; H^\beta)$ of (1.1) such that $u(t)$ approaches $u_\pm(t)$ in $X^{\beta,0}$ as $t \rightarrow \pm\infty$, where $u_\pm(t)$ are solutions of the linear Klein-Gordon equation with initial data (f_\pm, g_\pm) , respectively. Moreover, as $\pm t$ large enough we have*

$$\|(u(t), \partial_t u(t)) - (u_\pm(t), \partial_t u_\pm(t))\|_{X^{\beta,0}} \leq C\langle t \rangle^{-\delta}$$

with $\delta = \frac{2n\beta}{n+2} - 2 > 0$.

(2). *For $(f_-, g_-) \in X^{\beta,1}$ with $\|(f_-, g_-)\|_{X^{\beta,1}} \leq \delta_0$, there uniquely exists a final state $(f_+, g_+) \in X^{\beta,0}$ and a solution $u(t) \in C(R; H^\beta)$ of (1.1) such that $u(t)$ approaches $u_\pm(t)$ in $X^{\beta,0}$ as $t \rightarrow \pm\infty$, where $u_\pm(t)$ are solutions of the linear Klein-Gordon equation with initial data (f_\pm, g_\pm) , respectively. Moreover, as $\pm t$ large enough we have*

$$\|(u(t), \partial_t u(t)) - (u_\pm(t), \partial_t u_\pm(t))\|_{X^{\beta,0}} \leq C\langle t \rangle^{-\delta}$$

with $\delta = \frac{2n\beta}{n+2} - 2 > 0$.

In this article we denote by $J_\varepsilon = \langle i\nabla \rangle x + i\varepsilon t \nabla$, $L_\varepsilon = i\partial_t - \varepsilon \langle i\nabla \rangle$ and $P = t\nabla + x\partial_t$ with $\varepsilon \in \{+, -\}$. For a given Banach space with norm $\|\cdot\|$ and a vector $v = (v^+, v^-)$, denote by

$$\begin{aligned} \|v\| &= \|v^+\| + \|v^-\|, \quad \|Pv\| = \|Pv^+\| + \|Pv^-\|, \\ \|Jv\| &= \|J_+v^+\| + \|J_-v^-\|, \quad \|Lv\| = \|L_+v^+\| + \|L_-v^-\|. \end{aligned}$$

We also denote by the space-time norm

$$\|\phi\|_{L_t^r(I, L_x^q)} = \|\|\phi(t)\|_{L_x^q(R^n)}\|_{L_t^r(I)},$$

where I is a bounded or unbounded time interval, and denote different positive constants by the same letter C .

The rest of the article is organized as follows. In Section 2 we give some preliminary calculations. Then Section 3 is devoted to the proof of Theorem 1.1.

2 Preliminaries

In this section, we prove some lemmas for our main results. Let $w^\varepsilon = i\partial_t \langle i\nabla \rangle^{-1} u - \varepsilon u$ with $\varepsilon \in \{+, -\}$. Then the Klein-Gordon equation (1.1) can be rewritten as a system of equations

$$\begin{cases} L_\varepsilon w^\varepsilon = \langle i\nabla \rangle^{-1} F_\gamma(u) \\ w^\varepsilon|_{t=0} = w_0^\varepsilon \end{cases} \quad (2.1)$$

where $L_\varepsilon = i\partial_t - \varepsilon \langle i\nabla \rangle$, $w_0^\varepsilon = i\langle i\nabla \rangle^{-1} g + \varepsilon f$. By the fact that

$$u = \frac{1}{2}(w^+ - w^-), \partial_t u = -\frac{i}{2}\langle i\nabla \rangle(w^+ + w^-),$$

we can rewrite the term $F_\gamma(u)$ as

$$F_\gamma(u) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} (V_\gamma * \overline{w^{\varepsilon_1}} w^{\varepsilon_2}) w^{\varepsilon_3}$$

with some constants $C_{\varepsilon_1 \varepsilon_2 \varepsilon_3}$. Denote $U_\varepsilon(t)\varphi = e^{-\varepsilon i \langle i\nabla \rangle t} \varphi$ and for given $T \in R$,

$$\Psi_\varepsilon[g] = \int_T^t U_\varepsilon(t - \tau) \langle i\nabla \rangle^{-1} g(\tau) d\tau,$$

Lemma 2.1 *Let $2 \leq q < \frac{2n}{n-2}$, $\frac{2}{r} = \frac{n}{2}(1 - \frac{2}{q})$. Then for any time interval I and for any given $T \in I$ the following estimates are true:*

$$\|\Psi_\varepsilon[g]\|_{L_t^r(I, L^q)} \leq \|g\|_{L_t^{r'}(I, H_{q'}^{2\mu-1})},$$

$$\|\Psi_\varepsilon[g]\|_{L_t^\infty(I, L^2)} \leq \|g\|_{L_t^{r'}(I, H_{q'}^{\mu-1})},$$

and

$$\|U_\varepsilon(t)\varphi\|_{L_t^r(I, L^q)} \leq \|\varphi\|_{H^\mu},$$

where $r' = \frac{r}{r-1}$, $q' = \frac{q}{q-1}$ and $\mu = \frac{1}{2}(1 + \frac{n}{2})(1 - \frac{2}{q})$.

The proof of Lemma 2.1 is based on the duality argument along with the $L^p - L^q$ time decay estimates. The similar result be found in [5].

Lemma 2.2 *Assume $2 \leq p < \frac{2n}{n-2}$ for $n \geq 3$ ($2 \leq p < +\infty$ for $n = 2$), denote by $\alpha = (1 + \frac{n}{2})(1 - \frac{2}{p})$. The estimate is valid*

$$\|\phi\|_{L^p} \leq C \langle t \rangle^{-\frac{n}{2}(1 - \frac{2}{p})} (\|\phi\|_{H^\alpha} + \|J_\varepsilon \phi\|_{H^{\alpha-1}}),$$

for all $t \in R$, provided that the right-hand side is finite.

This lemma comes from Lemma 2.1 in [5] and the fact that $\|\phi\|_{L^p} \leq C\|\phi\|_{H^\alpha}$ when $p \geq 2$.

Lemma 2.3 *Assume $|V_\gamma(x)| \leq |x|^{-\gamma}$ with $0 < \gamma < n$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}$.*

(1). *For $1 < r < +\infty$, $1 < p_1, p_2 < +\infty$ and $p_3 > r$ satisfying $1 + \frac{1}{r} = \frac{2}{n} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$, we have*

$$\|(V_\gamma * \overline{w^{\varepsilon_1}} w^{\varepsilon_2}) w^{\varepsilon_3}\|_{L^r} \leq \|w^{\varepsilon_1}\|_{L^{p_1}} \|w^{\varepsilon_2}\|_{L^{p_2}} \|w^{\varepsilon_3}\|_{L^{p_3}}.$$

(2). For $\rho > 0$, $1 < r < +\infty$, $1 < p_{jk} < +\infty$ for $j, k \in \{1, 2\}$ and $p_{13}, p_{23} > r$ satisfying $1 + \frac{1}{r} = \frac{\gamma}{n} + \frac{1}{p_{j1}} + \frac{1}{p_{j2}} + \frac{1}{p_{j3}}$, we have

$$\begin{aligned} \|(V_\gamma * \overline{w^{\varepsilon_1}} w^{\varepsilon_2}) w^{\varepsilon_3}\|_{H_r^\rho} &\leq \|w^{\varepsilon_1}\|_{H_{p_{11}}^\rho} \|w^{\varepsilon_2}\|_{L^{p_{12}}} \|w^{\varepsilon_3}\|_{L^{p_{13}}} + \|w^{\varepsilon_1}\|_{L^{p_{12}}} \|w^{\varepsilon_2}\|_{H_{p_{11}}^\rho} \|w^{\varepsilon_3}\|_{L^{p_{13}}} \\ &\quad + \|w^{\varepsilon_1}\|_{L^{p_{21}}} \|w^{\varepsilon_2}\|_{L^{p_{22}}} \|w^{\varepsilon_3}\|_{H_{p_{23}}^\rho} \end{aligned}$$

Proof. To prove (1), put $\frac{1}{p_4} = \frac{1}{r} - \frac{1}{p_3}$. By the Hölder inequality and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} \|(V_\gamma * \overline{w^{\varepsilon_1}} w^{\varepsilon_2}) w^{\varepsilon_3}\|_{L^r} &\leq \|V_\gamma * \overline{w^{\varepsilon_1}} w^{\varepsilon_2}\|_{L^{p_4}} \|w^{\varepsilon_3}\|_{L^{p_3}} \\ &\leq \|w^{\varepsilon_1}\|_{L^{p_1}} \|w^{\varepsilon_2}\|_{L^{p_2}} \|w^{\varepsilon_3}\|_{L^{p_3}} \end{aligned}$$

since $1 + \frac{1}{p_4} = \frac{\gamma}{n} + \frac{1}{p_1} + \frac{1}{p_2}$.

To prove (2), we set $\frac{1}{r} = \frac{1}{p_{14}} + \frac{1}{p_{13}}$ and $\frac{1}{r} = \frac{1}{p_{24}} + \frac{1}{p_{23}}$. For $\rho > 0$, the generalized Hölder inequality in [6] implies

$$\|(V_\gamma * \overline{w^{\varepsilon_1}} w^{\varepsilon_2}) w^{\varepsilon_3}\|_{H_r^\rho} \leq \|V_\gamma * \overline{w^{\varepsilon_1}} w^{\varepsilon_2}\|_{H_{p_{14}}^\rho} \|w^{\varepsilon_3}\|_{L^{p_{13}}} + \|V * \overline{w^{\varepsilon_1}} w^{\varepsilon_2}\|_{L^{p_{24}}} \|w^{\varepsilon_3}\|_{H_{p_{23}}^\rho}$$

By the generalized Hölder inequality and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} \|V_\gamma * \overline{w^{\varepsilon_1}} w^{\varepsilon_2}\|_{H_{p_{14}}^\rho} &\leq \|V_\gamma * \langle i\nabla \rangle^\rho (\overline{w^{\varepsilon_1}} w^{\varepsilon_2})\|_{L^{p_{14}}} \leq \|\langle i\nabla \rangle^\rho (\overline{w^{\varepsilon_1}} w^{\varepsilon_2})\|_{L^{p_{15}}} \\ &\leq \|w^{\varepsilon_1}\|_{H_{p_{11}}^\rho} \|w^{\varepsilon_2}\|_{L^{p_{12}}} + \|w^{\varepsilon_2}\|_{H_{p_{11}}^\rho} \|w^{\varepsilon_1}\|_{L^{p_{12}}} \end{aligned}$$

since $1 + \frac{1}{p_{14}} = \frac{\gamma}{n} + \frac{1}{p_{15}}$ and $\frac{1}{p_{15}} = \frac{1}{p_{11}} + \frac{1}{p_{12}}$. Similarly we have

$$\|V_\gamma * \overline{w^{\varepsilon_1}} w^{\varepsilon_2}\|_{L^{p_{24}}} \leq \|w^{\varepsilon_1}\|_{L^{p_{21}}} \|w^{\varepsilon_2}\|_{L^{p_{22}}}.$$

□

3 Proof of Theorem 1.1

For $1 < \gamma < \min\{\frac{2(n+1)}{n+2}, \frac{3n-2}{n+2}\}$, we choose

$$\frac{(n+2)(\gamma+1)}{4n} + \frac{1}{2} < \beta < \frac{(n+2)(\gamma+1)}{2n}, q = \left(\frac{2\beta}{n+2} + \frac{1}{2} - \frac{\gamma+1}{n} \right)^{-1},$$

They satisfy

$$1 \leq \beta \leq 2, 2 < q < \frac{2n}{n+2(1-\gamma)}, 1 < \gamma < \frac{3n\beta}{n+2}.$$

Let $\mu = \frac{1}{2}(1 + \frac{n}{2})(1 - \frac{2}{q})$, we also have

$$\mu + \beta - 2 \leq 0, \mu \leq \beta - 1, \text{ and } 0 < \mu \leq \frac{1}{2}.$$

Let r, p and s be chosen as

$$\frac{2}{r} = \frac{n}{2}(1 - \frac{2}{q}), \frac{2}{p} + \frac{\gamma}{n} = 2 - \frac{2}{q}, \frac{2}{s} = 1 - \frac{2}{r}.$$

The proof of Theorem 1.1(1). Introduce the function space

$$X = \{v = (v^+, v^-) \in C(R; (L^2(R^n))^2); \quad \|v\|_X < +\infty\}$$

with the norm

$$\begin{aligned} \|v\|_X &= \|v\|_{L_t^\infty(R, H^\beta)} + \|v\|_{L_t^r(R, H_q^{\beta-\mu})} + \|\partial_t v\|_{L_t^\infty(R, H^{\beta-1})} + \|\partial_t v\|_{L_t^r(R, L^q)} \\ &\quad + \|Pv\|_{L_t^\infty(R, H^{\beta-1})} + \|Pv\|_{L_t^r(R, L^q)} + \|Jv\|_{L_t^\infty(R, H^{\beta-1})}. \end{aligned}$$

Denote by X_ρ a ball of a radius $\rho > 0$ with a center in the origin in the space X . Let us consider the linearized version of (2.1)

$$\begin{cases} L_\varepsilon w^\varepsilon = \langle i\nabla \rangle^{-1} F_\gamma(v) \\ w^\varepsilon|_{t=0} = w_0^\varepsilon \end{cases} \quad (3.1)$$

with a given vector $v = (v^+, v^-) \in X_\rho$, where

$$F_\gamma(v) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} (V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}$$

with some given constants $C_{\varepsilon_1 \varepsilon_2 \varepsilon_3}$. The integration of the linearized Cauchy problem (3.1) with respect to time yields

$$w^\varepsilon = U_\varepsilon(t) w_0^\varepsilon + \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} \Psi_\varepsilon((V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}). \quad (3.2)$$

Taking the $L_t^\infty(R; H^\beta)$ -norm of (3.2), applying the Hölder inequality, Lemma 2.1 and Lemma 2.3, we find

$$\begin{aligned} \|w^\varepsilon\|_{L_t^\infty(R; H^\beta)} &\leq \|U_\varepsilon(t) w_0^\varepsilon\|_{L_t^\infty(R; H^\beta)} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|\Psi_\varepsilon((V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3})\|_{L_t^\infty(R; H^\beta)} \\ &\leq \|w_0^\varepsilon\|_{H^\beta} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|(V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}\|_{L_t^{r'}(R; H_q^{\beta+\mu-1})} \\ &\leq \|w_0\|_{H^\beta} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \left\| \|v^{\varepsilon_1}\|_{H_q^{\beta+\mu-1}} \|v^{\varepsilon_2}\|_{L^p} \|v^{\varepsilon_3}\|_{L^p} \right\|_{L_t^{r'}(R)} \\ &\leq \|w_0\|_{H^\beta} + C \|v\|_{L_t^r(R; H_q^{\beta-\mu})} \|v\|_{L_t^s(R; L^p)}^2 \\ &\leq \|w_0\|_{H^\beta} + C \rho \|v\|_{L_t^s(R; L^p)}^2 \end{aligned} \quad (3.3)$$

since $p > 2 > q'$, $q > 2 > q'$, $\mu \leq \frac{1}{2}$ and $2 - \frac{2}{q} = \frac{\gamma}{n} + \frac{2}{p}$. Similarly, taking the $L_t^r(R; H_q^{\beta-\mu})$ we obtain

$$\begin{aligned} \|w^\varepsilon\|_{L_t^r(R; H_q^{\beta-\mu})} &\leq \|U_\varepsilon(t) w_0^\varepsilon\|_{L_t^r(R; H_q^{\beta-\mu})} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|\Psi_\varepsilon((V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3})\|_{L_t^r(R; H_q^{\beta-\mu})} \\ &\leq \|w_0^\varepsilon\|_{H^\beta} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|(V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}\|_{L_t^{r'}(R; H_q^{\beta+\mu-1})} \\ &\leq \|w_0^\varepsilon\|_{H^\beta} + C \|v\|_{L_t^r(R; H_q^{\beta+\mu-1})} \|v\|_{L_t^s(R; L^p)}^2 \\ &\leq \|w_0\|_{H^\beta} + C \rho \|v\|_{L_t^s(R; L^p)}^2 \end{aligned} \quad (3.4)$$

since $\mu \leq \frac{1}{2}$, $p > 2 > q'$, $q > 2 > q'$ and $2 - \frac{2}{q} = \frac{\gamma}{n} + \frac{2}{p}$. Applying the operator ∂_t to (3.1) we deduce that $\partial_t w^\varepsilon$ satisfies the following system

$$\begin{cases} L_\varepsilon \partial_t w^\varepsilon = \langle i\nabla \rangle^{-1} \partial_t F_\gamma(v) \\ \partial_t w^\varepsilon|_{t=0} = -i\varepsilon \langle i\nabla \rangle w_0^\varepsilon - i \langle i\nabla \rangle^{-1} F_\gamma(v)|_{t=0} \end{cases}$$

with

$$F_\gamma(v) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} (V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}.$$

Then by integrating with respect to time,

$$\partial_t w^\varepsilon = U_\varepsilon(t)(\partial_t w^\varepsilon|_{t=0}) + \Psi_\varepsilon(\partial_t F_\gamma(v)).$$

Taking the $L_t^\infty(R; H^{\beta-1})$ -norm and $L_t^r(R, L^q)$ -norm, applying the Hölder inequality and Lemma 2.1 we find that, since $\beta \geq 1$, $\mu \leq \beta - 1$ and $\mu + \beta - 2 \leq 0$,

$$\begin{aligned} & \|\partial_t w^\varepsilon\|_{L_t^\infty(R; H^{\beta-1})} + \|\partial_t w^\varepsilon\|_{L_t^r(R; L^q)} \\ & \leq \|\partial_t w^\varepsilon|_{t=0}\|_{H^{\beta-1}} + \|\partial_t F_\gamma(v)\|_{L_t^{r'}(R, H_q^{\mu+\beta-2})} \\ & \leq \|\partial_t w^\varepsilon|_{t=0}\|_{H^{\beta-1}} \\ & \quad + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|(V_\gamma * (\overline{\partial_t v^{\varepsilon_1}} v^{\varepsilon_2} + \overline{v^{\varepsilon_1}} \partial_t v^{\varepsilon_2})) v^{\varepsilon_3} + (V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) \partial_t v^{\varepsilon_3}\|_{L_t^{r'}(R; L^{q'})} \\ & \leq \|\partial_t w^\varepsilon|_{t=0}\|_{H^{\beta-1}} + C \|\partial_t v\|_{L_t^r(R, L^q)} \|v\|_{L_t^s(R, L^p)}^2 \\ & \leq \|\partial_t w^\varepsilon|_{t=0}\|_{H^{\beta-1}} + C \rho \|v\|_{L_t^s(R, L^p)}^2 \end{aligned}$$

On the other hand, we have

$$\|\partial_t w^\varepsilon|_{t=0}\|_{H^{\beta-1}} \leq \|w_0^\varepsilon\|_{H^\beta} + \|F_\gamma(v)\|_{L_t^\infty(R, H^{\beta-2})},$$

and for $p_1 > 2$ satisfying $\frac{3}{2} = \frac{\gamma}{n} + \frac{3}{p_1}$,

$$\begin{aligned} \|F_\gamma(v)\|_{L_t^\infty(R, H^{\beta-2})} & \leq C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|(V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}\|_{L_t^\infty(R; L^2)} \\ & \leq C \|v\|_{L_t^\infty(R; L^{p_1})}^3 \leq C \|v\|_{L_t^\infty(R; H^\beta)}^3 \leq C \rho^3 \end{aligned}$$

since $\beta \leq 2, \gamma \leq 3\beta$ and $\|v\|_{L^{p_1}} \leq C \|v\|_{H^\beta}$. Then

$$\|\partial_t w^\varepsilon\|_{L_t^\infty(R; H^{\beta-1})} + \|\partial_t w^\varepsilon\|_{L_t^r(R; L^q)} \leq C \|w_0\|_{H^\beta} + C \rho^3 + C \rho \|v\|_{L_t^s(R, L^p)}^2. \quad (3.5)$$

Notice that $P = t \nabla + x \partial_t$, $J_\varepsilon = \langle i\nabla \rangle x + i\varepsilon t \nabla$ and $L_\varepsilon = i \partial_t - \varepsilon \langle i\nabla \rangle$. We get

$$\begin{aligned} J_\varepsilon &= i\varepsilon P - \varepsilon L_\varepsilon, [L_\varepsilon, P] = -i\varepsilon \langle i\nabla \rangle^{-1} \nabla L_\varepsilon, \\ [x, \langle i\nabla \rangle] &= \langle i\nabla \rangle^{-1} \nabla, [P, \langle i\nabla \rangle^{-1}] = \langle i\nabla \rangle^{-3} \nabla \partial_t \end{aligned}$$

and

$$P((V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}) = (V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) P(v^{\varepsilon_3}) + (t \nabla V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}.$$

Applying the operator P to (3.1) yields

$$\begin{cases} L_\varepsilon Pw^\varepsilon = i\varepsilon \langle i\nabla \rangle^{-2} \nabla F_\gamma(v) - \langle i\nabla \rangle^{-1} P F_\gamma(v) - \langle i\nabla \rangle^{-3} \nabla \partial_t F_\gamma(v) \\ Pw^\varepsilon|_{t=0} = x \partial_t w^\varepsilon|_{t=0} = x(-i\varepsilon \langle i\nabla \rangle w_0^\varepsilon - i \langle i\nabla \rangle^{-1} F_\gamma(v)|_{t=0}) \end{cases}$$

with

$$P F_\gamma(v) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} (V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) P v^{\varepsilon_3} + (t \nabla V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}.$$

Integrating with respect to time, we get

$$Pw^\varepsilon = U_\varepsilon(t)(Pw^\varepsilon|_{t=0}) - \Psi_\varepsilon(i\varepsilon \langle i\nabla \rangle^{-1} \nabla F_\gamma(v)) + \Psi_\varepsilon(P F_\gamma(v)) + \Psi_\varepsilon(\langle i\nabla \rangle^{-2} \nabla \partial_t F_\gamma(v)). \quad (3.6)$$

Taking the $L_t^\infty(R; H^{\beta-1})$ -norm and the $L_t^r(R, L^q)$ -norm of (3.6), applying the Hölder inequality and Lemma 2.1, we find

$$\begin{aligned} & \|Pw^\varepsilon\|_{L_t^\infty(R; H^{\beta-1})} + \|Pw^\varepsilon\|_{L_t^r(R, L^q)} \\ & \leq \|Pw^\varepsilon|_{t=0}\|_{H^{\beta-1}} + \|\langle i\nabla \rangle^{-1} \nabla F_\gamma\|_{L_t^{r'}(R, L^{q'})} \\ & \quad + \|P F_\gamma(v)\|_{L_t^{r'}(R, L^{q'})} + \|\langle i\nabla \rangle^{-2} \nabla \partial_t F_\gamma(v)\|_{L_t^{r'}(R, L^{q'})} \\ & \leq \|Pw^\varepsilon|_{t=0}\|_{H^{\beta-1}} + \|F_\gamma(v)\|_{L_t^{r'}(R, L^{q'})} \\ & \quad + \|P F_\gamma(v)\|_{L_t^{r'}(R, L^{q'})} + \|\partial_t F_\gamma(v)\|_{L_t^{r'}(R, L^{q'})} \end{aligned} \quad (3.7)$$

since $\beta \geq 1$ and $\mu + \beta - 2 \leq 0$ and $\mu \leq \beta - 1$. As in the proof of (3.5) we deduce

$$\begin{aligned} & \|F_\gamma(v)\|_{L_t^{r'}(R, L^{q'})} + \|\partial_t F_\gamma(v)\|_{L_t^{r'}(R, L^{q'})} \\ & \leq C \|v\|_{L_t^r(R, L^q)} \|v\|_{L_t^s(R, L^p)}^2 + C \|\partial_t v\|_{L_t^r(R, L^q)} \|v\|_{L_t^s(R, L^p)}^2 \leq C \rho \|v\|_{L_t^s(R, L^p)}^2. \end{aligned} \quad (3.8)$$

Let $p_3 > 2$ and $s_3 > 2$ satisfy

$$\frac{3}{2} - \frac{1}{q} = \frac{\gamma+1}{n} + \frac{2}{p_3}, \quad 1 - \frac{1}{r} = \frac{2}{s_3}.$$

The Hölder inequality and Lemma 2.3 imply

$$\begin{aligned} & \|P F_\gamma(v)\|_{L_t^{r'}(R, L^{q'})} \\ & \leq C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \left[\|(V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) P v^{\varepsilon_3}\|_{L_t^{r'}(R, L^{q'})} + \|(t \nabla V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}\|_{L_t^{r'}(R, L^{q'})} \right] \\ & \leq C \|Pv\|_{L_t^r(R, L^q)} \|v\|_{L_t^s(R, L^p)}^2 + C \|v\|_{L_t^\infty(R, L^2)} \|t^{1/2} v\|_{L_t^{s_3}(R, L^{p_3})}^2 \\ & \leq C \rho \|v\|_{L_t^s(R, L^p)}^2 + C \rho \|t^{1/2} v\|_{L_t^{s_3}(R, L^{p_3})}^2, \end{aligned} \quad (3.9)$$

here we use the condition $\|\nabla V_\gamma\| \leq C|x|^{-(\gamma+1)}$. By Lemma 2.2 we have

$$\begin{aligned} \|v\|_{L_t^s(R, L^p)} & \leq C \|\langle t \rangle^{-\frac{n}{2}(1-\frac{1}{p})} (\|v\|_{H^\alpha} + \|Jv\|_{H^{\alpha-1}})\|_{L_t^s(R)} \\ & \leq C \left(\|v\|_{L_t^\infty(R, H^\beta)} + \|Jv\|_{L_t^\infty(R, H^{\beta-1})} \right) \leq C \rho \end{aligned} \quad (3.10)$$

since $\alpha = (1 + \frac{n}{2})(1 - \frac{2}{p}) \leq \beta$ and $\frac{n}{2}(1 - \frac{2}{p}) > \frac{1}{s}$. Similarly,

$$\begin{aligned} \|t^{1/2}v\|_{L_t^{\varepsilon_3}(R, L^{p_3})} &\leq C \|\langle t \rangle^{-\frac{n}{2}(1 - \frac{1}{p_3}) + \frac{1}{2}} (\|v\|_{H^{\alpha_3}} + \|Jv\|_{H^{\alpha_3-1}})\|_{L_t^{\varepsilon_3}(R)} \\ &\leq C \left(\|v\|_{L_t^\infty(R, H^\beta)} + \|Jv\|_{L_t^\infty(R, H^{\beta-1})} \right) \leq C\rho, \end{aligned} \quad (3.11)$$

since $\alpha_3 = (1 + \frac{n}{2})(1 - \frac{2}{p_3}) \leq \beta$ and $\frac{n}{2}(1 - \frac{2}{p_3}) > \frac{1}{s_3}$. Then we obtain, from (3.7)-(3.11),

$$\|Pw^\varepsilon\|_{L_t^\infty(R, H^{\beta-1})} + \|Pw^\varepsilon\|_{L_t^r(R, L^q)} \leq \|Pw^\varepsilon|_{t=0}\|_{H^{\beta-1}} + C\rho^3, \quad (3.12)$$

$$\|w^\varepsilon\|_{L_t^\infty(R, H^\beta)} + \|w^\varepsilon\|_{L_t^r(R, H_q^{\beta-\mu})} \leq \|w_0^\varepsilon\|_{H^\beta} + C\rho^3, \quad (3.13)$$

$$\|\partial_t w^\varepsilon\|_{L_t^\infty(R, H^{\beta-1})} + \|\partial_t w^\varepsilon\|_{L_t^r(R, L^q)} \leq \|w_0^\varepsilon\|_{H^\beta} + C\rho^3, \quad (3.14)$$

To estimate the term $\|Pw^\varepsilon|_{t=0}\|_{H^{\beta-1}}$, we give some estimates. It follows from the Sobolev embedding theorem that

$$\begin{aligned} \|F_\gamma(v)\|_{L_t^\infty(R, L^2)} &\leq C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|(V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2}) v^{\varepsilon_3}\|_{L_t^\infty(R, L^2)} \\ &\leq \|v\|_{L_t^\infty(R, L^{p_5})}^3 \leq C \|v\|_{L_t^\infty(R, H^\beta)}^3 \leq C\rho^3, \end{aligned} \quad (3.15)$$

where $p_5 = \frac{6n}{3n-2\gamma}$, which satisfies $p_5 \leq \frac{2n}{n-2\beta}$ because of $\gamma \leq 3\beta$. Using the relation $x = \langle i\nabla \rangle^{-1} J_\varepsilon - i\varepsilon t \langle i\nabla \rangle^{-1} \nabla$ we deduce

$$\begin{aligned} \|xF_\gamma(v)\|_{L_t^\infty(R, L^2)} &\leq C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|(V_\gamma * \overline{v^{\varepsilon_1}} v^{\varepsilon_2})(xv^{\varepsilon_3})\|_{L_t^\infty(R, L^2)} \\ &\leq C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \left\| \|v^{\varepsilon_1}\|_{L^{p_4}} \|v^{\varepsilon_2}\|_{L^{p_4}} \left(\|\langle i\nabla \rangle^{-1} J_\varepsilon v^{\varepsilon_3}\|_{L^{p_4}} + t \|\langle i\nabla \rangle^{-1} \nabla v^{\varepsilon_3}\|_{L^{p_4}} \right) \right\|_{L_t^\infty(R)} \\ &\leq C \|v\|_{L_t^\infty(R, L^{p_4})}^2 \|\langle i\nabla \rangle^{-1} Jv\|_{L_t^\infty(R, L^{p_4})} + C \|t^{1/3}v\|_{L_t^\infty(R, L^{p_4})}^3 \\ &\leq C \|v\|_{L_t^\infty(R, H^\beta)}^2 \|Jv\|_{L_t^\infty(R, H^{\beta-1})} + C \left(\|v\|_{L_t^\infty(R, H^\beta)} + \|Jv\|_{L_t^\infty(R, H^{\beta-1})} \right)^3 \\ &\leq C \left(\rho + \|Jv\|_{L_t^\infty(R, H^{\beta-1})} \right)^3 \leq C\rho^3, \end{aligned} \quad (3.16)$$

where $p_4 = \frac{6n}{3n-2\gamma}$, which satisfies

$$2 < p_4 \leq \frac{2n}{n-2\beta}, \quad \frac{n}{2}(1 - \frac{2}{p_4}) \geq \frac{1}{3}, \quad (1 + \frac{n}{2})(1 - \frac{2}{p_4}) \leq \beta$$

because of $1 < \gamma \leq \frac{3n\beta}{n+2}$. Using the relation $[\langle i\nabla \rangle^{\beta-1}, x] = -(\beta-1)\langle i\nabla \rangle^{\beta-3}\nabla$ we deduce

$$\begin{aligned} \|Pw^\varepsilon|_{t=0}\|_{H^{\beta-1}} &\leq \|x\langle i\nabla \rangle w_0^\varepsilon\|_{H^{\beta-1}} + \|x\langle i\nabla \rangle^{-1} F_\gamma(v)\|_{L_t^\infty(R, H^{\beta-1})} \\ &\leq \|x\langle i\nabla \rangle w_0^\varepsilon\|_{H^{\beta-1}} + \|\langle i\nabla \rangle^{-1} xF_\gamma(v)\|_{L_t^\infty(R, H^{\beta-1})} + \|\langle i\nabla \rangle^{-3} \nabla F_\gamma(v)\|_{L_t^\infty(R, H^{\beta-1})} \\ &\leq \|\langle i\nabla \rangle^{\beta-1} x\langle i\nabla \rangle w_0^\varepsilon\|_{L^2} + C \|xF_\gamma(v)\|_{L_t^\infty(R, L^2)} + C \|F_\gamma(v)\|_{L_t^\infty(R, L^2)} \\ &\leq \|\langle x \rangle \langle i\nabla \rangle^\beta w_0^\varepsilon\|_{L^2} + C\rho^3 + C \left(\rho + \|Jv\|_{L_t^\infty(R, H^{\beta-1})} \right)^3 \\ &\leq \|w_0\|_{H^{\beta,1}} + C\rho^3, \end{aligned}$$

which, combining with (3.12), yields

$$\|Pw^\varepsilon\|_{L_t^\infty(R;H^{\beta-1})} + \|Pw^\varepsilon\|_{L_t^1(R,L^q)} \leq \|w_0\|_{H^{\beta,1}} + C\rho^3. \quad (3.17)$$

Notice that

$$[L_\varepsilon, x] = -\varepsilon\langle i\nabla \rangle^{-1}\nabla, [x, \langle i\nabla \rangle^{-1}] = -\langle i\nabla \rangle^{-3}\nabla.$$

Then we deduce that xw^ε satisfies

$$L_\varepsilon(xw^\varepsilon) = -\varepsilon\langle i\nabla \rangle^{-1}\nabla w^\varepsilon - \langle i\nabla \rangle^{-1}(xF_\gamma(v)) + \langle i\nabla \rangle^{-1}\nabla F_\gamma(v).$$

Using $J_\varepsilon = i\varepsilon P - \varepsilon L_\varepsilon x$ and (3.13) yields

$$\|J_\varepsilon w^\varepsilon\|_{L_t^\infty(R,H^{\beta-1})} \leq \|Pw^\varepsilon\|_{L_t^\infty(R,H^{\beta-1})} + \|L_\varepsilon(xw^\varepsilon)\|_{L_t^\infty(R,H^{\beta-1})},$$

with

$$\begin{aligned} & \|L_\varepsilon(xw^\varepsilon)\|_{L_t^\infty(R,H^{\beta-1})} \\ & \leq \|w^\varepsilon\|_{L_t^\infty(R,H^{\beta-1})} + \|\langle i\nabla \rangle^{-2}F_\gamma(v)\|_{L_t^\infty(R,H^{\beta-1})} + \|\langle i\nabla \rangle^{-1}(xF_\gamma(v))\|_{L_t^\infty(R,H^{\beta-1})} \\ & \leq \|w^\varepsilon\|_{L_t^\infty(R,H^\beta)} + \|F_\gamma(v)\|_{L_t^\infty(R,L^2)} + \|xF_\gamma(v)\|_{L_t^\infty(R,L^2)} \leq C\|w_0\|_{H^\beta} + C\rho^3. \end{aligned}$$

Then we get

$$\|J_\varepsilon w^\varepsilon\|_{L_t^\infty(R,H^{\beta-1})} \leq C\|w_0\|_{H^{\beta,1}} + C\rho^3. \quad (3.18)$$

A combination of (3.12) with (3.13), (3.14), (3.17) and (3.18) yields

$$\|w\|_X \leq C\|w_0\|_{H^{\beta,1}} + C\rho^3. \quad (3.19)$$

Therefore the map $M : w = M(v)$ defined by the problem (3.1), transforms a ball X_ρ with a small radius $\rho = C\|w^0\|_{H^{\beta,1}}$ into itself. Denote $\tilde{w} = M(\tilde{v})$, then in the same way as in the proof of (3.19) we have

$$\|M(v) - M(\tilde{v})\|_X \leq C\rho^2\|v - \tilde{v}\|_X.$$

Thus M is a contraction mapping in X_ρ and so there exists a unique solution $w = M(w)$ of (3.1) if the norm $\|w^0\|_{H^{\beta,1}}$ is small enough.

To prove the asymptotic of the solution $w(t, x)$, we use the equation, for $|t| > |t'|$,

$$U_\varepsilon(-t)w^\varepsilon(t) - U_\varepsilon(-t')w^\varepsilon(t') = \int_{t'}^t U_\varepsilon(-\tau)\langle i\nabla \rangle^{-1}F_\gamma(w(\tau))d\tau.$$

Taking the H^β -norm of this equation, using the similar proof of (3.3) and (3.4), we deduce

$$\|U_\varepsilon(-t)w^\varepsilon(t) - U_\varepsilon(-t')w^\varepsilon(t')\|_{H^\beta} \leq C\rho^2\langle t' \rangle^{-\delta}$$

with $\delta = \frac{2n\beta}{n+2} - 2 > 0$, since we have $\|w\|_X \leq \rho$ and

$$\|\langle t \rangle^{-\frac{n}{2}(1-\frac{1}{p})}\|_{L_t^2([t',t])}^2 \leq C\langle t' \rangle^{-\delta}.$$

Then there uniquely exist final states $w_\pm^\varepsilon \in H^\beta$ satisfying, for $\pm t$ large enough,

$$\|w^\varepsilon(t) - U_\varepsilon(t)w_\pm^\varepsilon\|_{H^\beta} \leq C\rho^2\langle t \rangle^{-\delta}.$$

Set $u(t) = \frac{1}{2}(w^+(t) - w^-(t))$, $f_{\pm}(x) = \frac{1}{2}(w_{\pm}^+ - w_{\pm}^-)$, $g_{\pm}(x) = -\frac{i}{2}\langle i\nabla \rangle(w_{\pm}^+ + w_{\pm}^-)$ and $u_{\pm}(t) = \frac{1}{2}(U_+(t)w_{\pm}^+ - U_-(t)w_{\pm}^-)$. Then $u(t)$ and $u_{\pm}(t)$ satisfy Theorem 1.1(1).

Proof of Theorem 1.1(2). For given $(f_-, g_-) \in X^{\beta,1}$ and $v = \{v^+, v^-\} \in X_{\rho}$, we consider the linearized version of the final state problem of (3.1)

$$\begin{cases} L_{\varepsilon}w^{\varepsilon} = -\langle i\nabla \rangle^{-1}F_{\gamma}(v) \\ \|U_{\varepsilon}(t)w^{\varepsilon} - w_{-}^{\varepsilon}(x)\|_{H^{\beta}} \rightarrow 0 \text{ as } t \rightarrow \infty \end{cases}$$

with $w_{-}^{\varepsilon}(x) = i\langle i\nabla \rangle^{-1}g_{-}(x) - \varepsilon f_{-}(x) \in H^{\beta,1}$. The integration with respect to time yields

$$w^{\varepsilon}(t) = U_{\varepsilon}(t)w_{-}^{\varepsilon} + \int_{-\infty}^t U_{\varepsilon}(t-\tau)\langle i\nabla \rangle^{-1}F_{\gamma}(v(\tau))d\tau.$$

In the same way as in the proof of Theorem 1.1(1), we find that, if $\|(f_-, g_-)\|_{X^{\beta,1}} \leq \rho$ small, there uniquely exists a global solution $w^{\varepsilon}(t) \in C(R, H^{\beta})$ and a final state $w_{+}^{\varepsilon} \in H^{\beta}$ such that, as $t \rightarrow +\infty$,

$$\|w^{\varepsilon}(t) - U_{\varepsilon}(t)w_{+}^{\varepsilon}\|_{H^{\beta}} \leq C\langle t \rangle^{-\delta}$$

with $\delta = \frac{2n\beta}{n+2} - 2 > 0$. Set $u(t) = \frac{1}{2}(w^+(t) - w^-(t))$, $f_{+}(x) = \frac{1}{2}(w_{+}^+ - w_{+}^-)$, $g_{+}(x) = -\frac{i}{2}\langle i\nabla \rangle(w_{+}^+ + w_{+}^-)$ and $u_{+}(t) = \frac{1}{2}(U_+(t)w_{+}^+ - U_-(t)w_{+}^-)$. Then $u(t)$ and $u_{+}(t)$ satisfy Theorem 1.1(2).

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